ExteriorProduct (∧)

`ExteriorProduct[x1, x2, …]`
denotes the exterior (or Wedge) product of the expressions `x1`, `x2`, …

Details

- The `ExteriorProduct` is `Listable`.
- `ExteriorProduct[x1,x2,…]` and `Wedge[x1,x2,…]` are equivalent input forms for constructing the expression `x1∧x2∧…` . They do not simplify or change the expression in any way.
- In *GrassmannAlgebra*, the `Wedge` operator has been endowed with the extra attribute `Flat`. This ensures that the exterior product acts associatively. For example `Wedge[x, Wedge[y, z]]` is automatically reduced to `Wedge[x, y, z]`.
- The wedge symbol ∧ may be entered by using the *CommonGrassmannOperations* palette or the sequence ESC^ESC.
- An exterior product □∧□ may be entered by using the *CommonGrassmannOperations* palette. Continued exterior products may be entered by continued clicking on the □∧□ button. For example, three successive clicks produces □∧□∧□∧□. The □ are placeholders in which expressions may be entered successively using the Tab key.
- For simplifying exterior products `ExpandAndSimplifyExteriorProducts` is faster than ⋆𝒢 and if the expression uses only basis vectors `GrassmannBreakout` is faster than either.

Examples (8)

Basic Examples (1)

In[1]:=
```
<< GrassmannCalculus`
```

Setting some Book preferences:

In[2]:=
```
⋆A; ⋆𝒫5;
```

These inputs are equivalent:

In[3]:=
```
{ExteriorProduct[x, y], Wedge[x, y], x∧y}
```

Out[3]=
```
{x∧y, x∧y, x∧y}
```

Because the exterior product is associative these inputs are also equivalent:

In[4]:=
```
{ExteriorProduct[x, y∧z], Wedge[Wedge[x, y], z], Wedge[x∧Wedge[y]∧z]}
```

Out[4]=
```
{x∧y∧z, x∧y∧z, x∧y∧z}
```

The exterior product is listable.

In[5]:=
```
{a, b, c} ∧ {x, y, z}
```
Out[5]=
```
{a ∧ x, b ∧ y, c ∧ z}
```

The following shows relative timings for various linear expansions of an exterior product. If the product contains only basis vectors and scalars `GrassmannBreakout` is the fastest.

In[6]:=
```
★𝓑5;
```
In[7]:=
```
product1 = (e1 + 2 e2 + 3 e3 + 4 e4) ∧ (-e1 + 3 e2 - e3 + e5) ∧ (e2 - 5 e3 + 2 e4 + e5) ∧ (e1 + 5 e2 - 3 e3 + 7 e5);
```
In[8]:=
```
product1 // GrassmannBreakout[Wedge, Automatic] // Timing
product1 // ExpandAndSimplifyExteriorProducts // Timing
product1 // ★𝒢 // Timing
```
Out[8]= $\{0.0468003,\ 216\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 - 144\, e_1 \wedge e_2 \wedge e_3 \wedge e_5 + 64\, e_1 \wedge e_2 \wedge e_4 \wedge e_5 + 184\, e_1 \wedge e_3 \wedge e_4 \wedge e_5 - 400\, e_2 \wedge e_3 \wedge e_4 \wedge e_5\}$

Out[8]= $\{0.764405,\ 216\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 - 144\, e_1 \wedge e_2 \wedge e_3 \wedge e_5 + 64\, e_1 \wedge e_2 \wedge e_4 \wedge e_5 + 184\, e_1 \wedge e_3 \wedge e_4 \wedge e_5 - 400\, e_2 \wedge e_3 \wedge e_4 \wedge e_5\}$

Out[8]= $\{3.75962,\ 216\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 - 144\, e_1 \wedge e_2 \wedge e_3 \wedge e_5 + 64\, e_1 \wedge e_2 \wedge e_4 \wedge e_5 + 184\, e_1 \wedge e_3 \wedge e_4 \wedge e_5 - 400\, e_2 \wedge e_3 \wedge e_4 \wedge e_5\}$

Exterior Products and Linear Dependence (1)

The exterior product is the foundation of Grassmann algebra. The exterior product represents the operation of constructing objects by extending the linear dimension of an initial object to a higher dimension. If the factors of an exterior product are not linearly independent then that operation fails and the result is zero. Therefore, in the context of linear algebra the exterior product is a test for linear dependence. If an exterior product evaluates to zero then its factors are linearly dependent.

Setting some Book preferences:

In[1]:=
```
★A; ★𝒫5;
```

As the Grassmann algebra application is designed, there are two conditions under which an exterior product will produce a zero result. The first condition is that all terms of the expanded product contain repeated 1-elements (or odd grade elements). This produces zero by exterior rule `★R[2, 1]`. This in turn is implied by exterior axiom 10.

In[2]:=
```
★R[2,1]
```
Out[2]=
```
___ ∧ x_ ∧ ___ ∧ x_ ∧ ___ /; OddGradeQ[x] :→ 0
```

So, as an example of linear dependence producing a zero result, we express `t` as a linear combination of the other 1-elements and evaluate by steps:

In[3]:=
```
p ∧ q ∧ r ∧ s ∧ t
% /. t → a p + b q + c r + d s
(% // ★ℰ) /. {★R[2, 4]}
% /. ★R[2, 1]
```
Out[3]=
```
p ∧ q ∧ r ∧ s ∧ t
```
Out[3]=
```
p ∧ q ∧ r ∧ s ∧ (a p + b q + c r + d s)
```
Out[3]=
```
a p ∧ q ∧ r ∧ s ∧ p + b p ∧ q ∧ r ∧ s ∧ q + c p ∧ q ∧ r ∧ s ∧ r + d p ∧ q ∧ r ∧ s ∧ s
```
Out[3]=
```
0
```

Each term has a repeat 1-element and thus the result is manifestly zero by the rules of the algebra.

The second condition is that the RawGrade of the exterior product is greater than the dimension of the underlying linear space. This rule is built-in to the definitions of GrassmannSimplify, ⋆𝒮.

In[4]:=
```
⋆D
p∧q∧r∧s∧t∧u∧v
{Grade[%], RawGrade[%]}
%% // ⋆𝒮
```

Out[4]= 6

Out[4]= p∧q∧r∧s∧t∧u∧v

Out[4]= {⋆0, 7}

Out[4]= 0

But what is the justification for the following results?

In[5]:=
```
ZeroQ[p∧q∧r∧s∧t∧u]
ZeroQ[⋆0∧e1∧e2∧e3∧e4∧e5]
```

Out[5]= False

Out[5]= False

In the second case we might say that the factors of the exterior product have been declared as basis elements for the space and are thus, by definition, independent. What about the first case? All we have formally declared about these symbols is that they are 1-elements. We have not declared anything about their dependence or independence and, in fact, above we made a substitution such that they were dependent.

In starting a derivation we should properly say: Let {p, q, r, s, t} be independent 1-elements. In the Grassmann algebra application this is assumed by *convention*. Any set of distinct undefined symbols (or symbolic expressions), up to the dimension of the linear space, are taken as being linearly independent. They are linearly dependent only if it is *manifest*, by using ZeroQ say, and this would only occur if we added additional definitions or used rules that established the dependence. Independence is the default.

The GrassmannAlgebra and GrassmannCalculus Applications considers any set of distinct 1-element symbols, without further definitions or values, up to the linear dimension of the space, to be independent.

We can formally state the property of the exterior product as

$$\forall_{k,0<k<\star n}\ (f_1 \wedge f_2 \wedge \ldots \wedge f_k) \wedge f_{k+1} == 0$$

if and only if f_{k+1} is expressly linearly dependent on the set $\{f_1, f_2, \ldots, f_k\}$ where ⋆n is the dimension of the underlying linear space.

Why Aren't Repeats of Even Grade Elements Zero? (1)

Often they are, and always in 2 and 3 dimensions. It seems they might be zero because if we dig into them won't there always be repeats of elements and won't that always produce zero? The answer is that this neglects the concept of simplicity and non-simplicity, which is discussed in Section 2.10 in the book.

To see how this works use a 4 dimensional vector space.

In[1]:=
```
⋆A; ⋆𝓑4;
```

In[2]:=
```
X = α_2 ∧ α_2
ZeroQ[X]
```

Out[2]= $\underset{2}{\alpha} \wedge \underset{2}{\alpha}$

Out[2]= False

To see why this can't be automatically taken as zero consider that $\underset{2}{\alpha}$ could be the sum of two terms. When the product is expanded we obtain four terms; two of them have 1-element repeats and reduce to zero, but the other two have four independent factors and sum to a non-zero term.

In[3]:=
```
X /. α_2 → u∧v + x∧y
% // ★ℰ
% // ★𝒮
```
Out[3]= $(u \wedge v + x \wedge y) \wedge (u \wedge v + x \wedge y)$

Out[3]= $u \wedge v \wedge u \wedge v + u \wedge v \wedge x \wedge y + x \wedge y \wedge u \wedge v + x \wedge y \wedge x \wedge y$

Out[3]= $2\, u \wedge v \wedge x \wedge y$

In a 3-dimensional space the product is zero because its factors exceed the underlying dimension and thus must contain duplicates.

In[4]:=
```
★A; ★ℬ3;
```
In[5]:=
```
X = α_2 ∧ α_2
ZeroQ[X]
```
Out[5]= $\underset{2}{\alpha} \wedge \underset{2}{\alpha}$

Out[5]= True

Examples of Exterior Products, Simplifications and Expansions (1)

In[1]:=
```
★A; ★𝒫5;
```
Using basis elements and simplifying:

In[2]:=
```
e3 ∧ e1 ∧ e5 ∧ e2
% // ★𝒮
```
Out[2]= $e_3 \wedge e_1 \wedge e_5 \wedge e_2$

Out[2]= $-(e_1 \wedge e_2 \wedge e_3 \wedge e_5)$

A point basis may include the Origin as a basis element.

In[3]:=
```
e2 ∧ ★O ∧ e1
% // ★𝒮
```
Out[3]= $e_2 \wedge ★\mathbb{O} \wedge e_1$

Out[3]= $★\mathbb{O} \wedge e_1 \wedge e_2$

Using vector sums, expanding (GrassmannExpand) and simplifying (GrassmannSimplify):

In[4]:=
```
(e1 + e3) ∧ (3 e2 - 5 e4 + e5) ∧ (a e1 + b e2 + c e3)
% // ★ℰ
% // ★𝒮
```
Out[4]= $(e_1 + e_3) \wedge (3\, e_2 - 5\, e_4 + e_5) \wedge (a\, e_1 + b\, e_2 + c\, e_3)$

Out[4]= $e_1 \wedge (3\, e_2) \wedge (a\, e_1) + e_1 \wedge (3\, e_2) \wedge (b\, e_2) + e_1 \wedge (3\, e_2) \wedge (c\, e_3) + e_1 \wedge (-5\, e_4) \wedge (a\, e_1) + e_1 \wedge (-5\, e_4) \wedge (b\, e_2) + e_1 \wedge (-5\, e_4) \wedge (c\, e_3) + e_1 \wedge e_5 \wedge (a\, e_1) + e_1 \wedge e_5 \wedge (b\, e_2) + e_1 \wedge e_5 \wedge (c\, e_3) + e_3 \wedge (3\, e_2) \wedge (a\, e_1) + e_3 \wedge (3\, e_2) \wedge (b\, e_2) + e_3 \wedge (3\, e_2) \wedge (c\, e_3) + e_3 \wedge (-5\, e_4) \wedge (a\, e_1) + e_3 \wedge (-5\, e_4) \wedge (b\, e_2) + e_3 \wedge (-5\, e_4) \wedge (c\, e_3) + e_3 \wedge e_5 \wedge (a\, e_1) + e_3 \wedge e_5 \wedge (b\, e_2) + e_3 \wedge e_5 \wedge (c\, e_3)$

Out[4]= $(-3\, a + 3\, c)\, e_1 \wedge e_2 \wedge e_3 + 5\, b\, e_1 \wedge e_2 \wedge e_4 - b\, e_1 \wedge e_2 \wedge e_5 + (-5\, a + 5\, c)\, e_1 \wedge e_3 \wedge e_4 + (a - c)\, e_1 \wedge e_3 \wedge e_5 - 5\, b\, e_2 \wedge e_3 \wedge e_4 + b\, e_2 \wedge e_3 \wedge e_5$

That could also be done in one step with GrassmannExpandAndSimplify:

In[5]:= $(e_1 + e_3) \wedge (3\, e_2 - 5\, e_4 + e_5) \wedge (a\, e_1 + b\, e_2 + c\, e_3)$ // ★$\mathcal{G}$

Out[5]= $(-3\, a + 3\, c)\, e_1 \wedge e_2 \wedge e_3 + 5\, b\, e_1 \wedge e_2 \wedge e_4 - b\, e_1 \wedge e_2 \wedge e_5 + (-5\, a + 5\, c)\, e_1 \wedge e_3 \wedge e_4 + (a - c)\, e_1 \wedge e_3 \wedge e_5 - 5\, b\, e_2 \wedge e_3 \wedge e_4 + b\, e_2 \wedge e_3 \wedge e_5$

Using symbolic vectors and simplifying:

In[6]:= $q \wedge p \wedge y \wedge x$
% // ★$\mathcal{S}$

Out[6]= $q \wedge p \wedge y \wedge x$

Out[6]= $p \wedge q \wedge x \wedge y$

Using graded symbols: (Ordering places odd grade elements before even grade elements.)

In[7]:= $\underset{2}{x} \wedge \underset{3}{\alpha} \wedge \underset{1}{b}$
% // ★$\mathcal{S}$

Out[7]= $\underset{2}{x} \wedge \underset{3}{\alpha} \wedge \underset{1}{b}$

Out[7]= $-\left(\underset{1}{b} \wedge \underset{3}{\alpha} \wedge \underset{2}{x}\right)$

Using a mixture of element types:

In[8]:= $x \wedge \underset{2}{\alpha} \wedge e_2$
% // ★$\mathcal{S}$

Out[8]= $x \wedge \underset{2}{\alpha} \wedge e_2$

Out[8]= $-\left(e_2 \wedge x \wedge \underset{2}{\alpha}\right)$

Using a multigraded symbol and expanding:

In[9]:= $x \wedge \underset{\{1,2\}}{\alpha}$
% // ComposeGradedForm
% // ★$\mathcal{E}$

Out[9]= $x \wedge \underset{\{1,2\}}{\alpha}$

Out[9]= $x \wedge \left(\underset{1}{\alpha} + \underset{2}{\alpha}\right)$

Out[9]= $x \wedge \underset{1}{\alpha} + x \wedge \underset{2}{\alpha}$

Simple Properties of Exterior Product (1)

In[1]:= ★A; ★$\mathcal{P}_5$;

Factoring of Scalars: (Using GrassmannSimplify)

In[2]:= $(a\, x) \wedge y == x \wedge (a\, y) == a\, (x \wedge y)$
% // ★$\mathcal{S}$

Out[2]= $(a\, x) \wedge y == x \wedge (a\, y) == a\, x \wedge y$

Out[2]= True

Distributivity: (Using GrassmannExpand)

```
In[3]:= x ∧ (y + z) == x ∧ y + x ∧ z
        % // ★ℰ
Out[3]= x ∧ (y + z) == x ∧ y + x ∧ z

Out[3]= True

In[4]:= (x + y) ∧ (z) == x ∧ z + y ∧ z
        % // ★ℰ
Out[4]= (x + y) ∧ z == x ∧ z + y ∧ z

Out[4]= True
```

Nilpotency: (Using GrassmannSimplify or GrassmanRule[2,1])

```
In[5]:= x ∧ x
        % // ★𝒮
Out[5]= x ∧ x

Out[5]= 0
```

This can also be affected using the following GrassmannRule:

```
In[6]:= GrassmannRule[2, 1] // Framed
        x ∧ x
        % /. GrassmannRule[2, 1]
Out[6]= ___ ∧ x_ ∧ ___ ∧ x_ ∧ ___ /; OddGradeQ[x] :→ 0

Out[6]= x ∧ x

Out[6]= 0
```

Antisymmetry: (Using GrassmannSimplify or GrassmannRule[2,9])

```
In[7]:= x ∧ y == - y ∧ x
        % // ★𝒮
Out[7]= x ∧ y == -y ∧ x

Out[7]= True

In[8]:= GrassmannRule[2, 9] // Framed
        x ∧ y == - (y ∧ x)
        MapAt[# /. GrassmannRule[2, 9] &, %, 2]
Out[8]= x_ ∧ y_ :→ (-1)^(RawGrade[x] RawGrade[y]) y ∧ x

Out[8]= x ∧ y == - (y ∧ x)

Out[8]= True
```

Generally, simplification can be achieved by using GrassmannExpandAndSimplify, ★𝒢.

Axioms of the Exterior Product (1)

```
In[1]:= ★A; ★𝒫5;
```

^ 1: The sum of m-elements is itself an m-element:

In[2]:= $\text{Grade}\left[\underset{m}{\alpha} + \underset{m}{\beta}\right]$

Out[2]= m

^ 2: Addition of m-elements is associative:

In[3]:= $\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) + \underset{m}{\gamma} == \underset{m}{\alpha} + \left(\underset{m}{\beta} + \underset{m}{\gamma}\right)$

Out[3]= True

^ 3: *m*-elements have an additive identity (zero element). However, a graded zero is not used in GrassmannAlgebra. Use a regular 0 instead.

In[4]:= $\underset{m}{\alpha} == \underset{m}{0} + \underset{m}{\alpha}$
$\% \text{ /. } \underset{m\_}{0} \to 0$

Out[4]= $\underset{m}{\alpha} == \underset{m}{0} + \underset{m}{\alpha}$

Out[4]= True

^ 4: *m*-elements have an additive inverse.

In[5]:= $\underset{m}{0} == \underset{m}{\alpha} + -\underset{m}{\alpha}$
$\% \text{ /. } \underset{m\_}{0} \to 0$

Out[5]= $\underset{m}{0} == 0$

Out[5]= True

^ 5: Addition of *m*-elements is commutative. (This is just *Mathematica* establishing a normal sort order.)

In[6]:= $\underset{m}{\alpha} + \underset{m}{\beta} == \underset{m}{\beta} + \underset{m}{\alpha}$

Out[6]= True

^ 6: The exterior product of an *m*-element and a *k*-element is an (*m*+*k*)-element.

In[7]:= $\text{Grade}\left[\underset{m}{\alpha} \wedge \underset{k}{\beta}\right]$

Out[7]= k + m

^ 7: The exterior product is associative. (This results from a Flat Wedge product.)

In[8]:= $\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \wedge \underset{j}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \wedge \underset{j}{\gamma}\right)$

Out[8]= True

$\wedge$8: There is a unit scalar which acts as an identity under the exterior product.

In[9]:=
```
α_m == 1 ∧ α_m // ★G
```
Out[9]= True

$\wedge$9: Non-zero scalars have inverses with respect to the exterior product. This is just the regular mathematical product inverse.

In[10]:=
```
a ∧ (1 / a)
% // ★G
```
Out[10]= $a \wedge \frac{1}{a}$

Out[10]= 1

$\wedge$10: The exterior product of elements of odd grade is anti-commutative. GrassmannAlgebra can only evaluate this for particular cases of k and m.

In[11]:=
```
step1 = α_m ∧ β_k == (-1)^(m k) β_k ∧ α_m
% /. {k → 1, m → 3}
% // ★G
```
Out[11]= $\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{k\,m}\, \underset{k}{\beta} \wedge \underset{m}{\alpha}$

Out[11]= $\underset{3}{\alpha} \wedge \underset{1}{\beta} == -\left(\underset{1}{\beta} \wedge \underset{3}{\alpha}\right)$

Out[11]= True

All other cases commute. For example:

In[12]:=
```
step1 /. {k → 1, m → 2}
% // ★G
```
Out[12]= $\underset{2}{\alpha} \wedge \underset{1}{\beta} == \underset{1}{\beta} \wedge \underset{2}{\alpha}$

Out[12]= True

The antisymmetry of 1-element products and nilpotency of like 1-elements follow from $\wedge$10. This is also implemented by GrassmannRule `{2, 1}` on the Rules Palettes.

$\wedge$11: Additive identities act as multiplicative zero elements under the exterior product. Again, GrassmannAlgebra everywhere uses a plain `0` instead of graded zeros.

In[13]:=
```
0_k ∧ α_m == 0_(k+m)
% /. 0_(m_) → 0
% // ★G
```
Out[13]= $\underset{k}{0} \wedge \underset{m}{\alpha} == \underset{k+m}{0}$

Out[13]= $0 \wedge \underset{m}{\alpha} == 0$

Out[13]= True

∧12: The exterior product is both left and right distributive under addition. Using `GrassmannExpand`:

In[14]:=
$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \wedge \underset{k}{\gamma} == \underset{m}{\alpha} \wedge \underset{k}{\gamma} + \underset{m}{\beta} \wedge \underset{k}{\gamma}$$
```
% // ★ℰ
```

Out[14]= $\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \wedge \underset{k}{\gamma} == \underset{m}{\alpha} \wedge \underset{k}{\gamma} + \underset{m}{\beta} \wedge \underset{k}{\gamma}$

Out[14]= True

In[15]:=
$$\underset{m}{\alpha} \wedge \left(\underset{k}{\beta} + \underset{k}{\gamma}\right) == \underset{m}{\alpha} \wedge \underset{k}{\beta} + \underset{m}{\alpha} \wedge \underset{k}{\gamma}$$
```
% // ★ℰ
```

Out[15]= $\underset{m}{\alpha} \wedge \left(\underset{k}{\beta} + \underset{k}{\gamma}\right) == \underset{m}{\alpha} \wedge \underset{k}{\beta} + \underset{m}{\alpha} \wedge \underset{k}{\gamma}$

Out[15]= True

Additional Rule for Exchanging Factors (1)

Determinants (1)

The Grassmann exterior product encapsulates the properties of determinants as developed in Section 2.7 of the book. Set a three dimensional space and define a matrix and calculate the determinant using the *Mathematica* `Det` operation.

In[1]:=
```
★𝓑3;
(mat = {{1, 2, 3}, {-2, 3, 0}, {0, 1, 2}}) // MatrixForm
Det[mat]
```

Out[1]//MatrixForm=
$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Out[1]= 8

Write rules defining 1-elements `{u, v, w}` on the rows of the matrix.

In[2]:=
```
elementRules = Thread[{u, v, w} → mat.Basis]
```

Out[2]= $\{u \to e_1 + 2 e_2 + 3 e_3,\ v \to -2 e_1 + 3 e_2,\ w \to e_2 + 2 e_3\}$

There exterior product gives the determinant times the unit n-basis.

In[3]:=
```
u ∧ v ∧ w
% /. elementRules
% // ★𝒢
```

Out[3]= $u \wedge v \wedge w$

Out[3]= $(e_1 + 2 e_2 + 3 e_3) \wedge (-2 e_1 + 3 e_2) \wedge (e_2 + 2 e_3)$

Out[3]= $8\, e_1 \wedge e_2 \wedge e_3$

Interchanging two rows switches the sign.

In[4]:=
```
v ∧ u ∧ w /. elementRules // ★𝒢
```

Out[4]= $-8\, e_1 \wedge e_2 \wedge e_3$

Duplicating a row gives zero.

In[5]:=
```
u∧u∧w /. elementRules // ⋆G
```
Out[5]= 0

Multiplying a row by a factor multiplies the determinant by the same factor.

In[6]:=
```
u∧(a v)∧w /. elementRules // ⋆G
```
Out[6]= $8\, a\, e_1 \wedge e_2 \wedge e_3$

The determinant is unchanged if to any row is added scalar multiples of other rows.

In[7]:=
```
(u + a v + b w)∧v∧w /. elementRules // ⋆G
```
Out[7]= $8\, e_1 \wedge e_2 \wedge e_3$

An exterior product, or determinant may be evaluated in a kind of Laplace expansion. Here we evaluate on row 1 and 3, and then on row 2 and combine the result in a new exterior product. However, we must include the Signature of the new row order that was introduced.

In[8]:=
```
step1 = u∧w /. elementRules // ⋆G
step2 = v /. elementRules // ⋆G
step1∧step2 Signature[{1, 3, 2}] // ⋆G
```
Out[8]= $e_1 \wedge e_2 + 2\, e_1 \wedge e_3 + e_2 \wedge e_3$

Out[8]= $-2\, e_1 + 3\, e_2$

Out[8]= $8\, e_1 \wedge e_2 \wedge e_3$

It will generally be more efficient to use the *Mathematica* linear solution routines but in small cases it may be more natural to use Grassmann procedures and to know how to transition between the two approaches.

See Also

RegressiveProduct ▪ **InteriorProduct** ▪ **CliffordProduct** ▪ **HypercomplexProduct** ▪ **GeneralizedProduct** ▪ **ExpandExteriorProducts** ▪ **ExpandAndSimplifyExteriorProducts** ▪ **SimplifyExteriorProducts** ▪ **OrderExterior** ▪ **ExteriorProductQ** ▪ **ExteriorFactorize** ▪ **ExteriorQuotient** ▪ **RegressiveToExterior** ▪ **InteriorToExterior** ▪ **NestedInteriorToExterior** ▪ **ToExteriorProducts** ▪ **ExteriorToInterior** ▪ **ExteriorToRegressive** ▪ **GrassmannBreakout**

Related Guides

- Expressions
- Exterior Product (∧)
- Grassmann Calculus